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Fisher information and Cramér-Rao bound for unknown systematic errors

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A R T I C L E I N F O

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A B S T R A C T

In order to understand the lower bound of achievable measurement uncertainties, the Cramér-Rao inequality is known to be an utmost useful tool. However, the calculation of the Cramér-Rao bound requires a known probability density function that describes the occurring stochastic process. For this reason, the Cramér-Rao bound is applied for determining the lower limit of the measurement uncertainty due to random errors. According to the international guide to the expression of uncertainty in measurement (GUM), unknown systematic errors shall be treated as random errors. This approach is adopted here to enhance the applicability of the Cramér-Rao bound for unknown systematic errors. As a key result, the concept of Fisher information and the Cramér-Rao bound is shown to be applicable also to unknown systematic errors, which is demonstrated for several examples. An unknown offset, an unknown linear drift and successive unknown linear drifts are investigated in detail as systematic errors. Each derived corresponding Fisher information shows a characteristic behavior with respect to the measurement time. In contrast to random errors with a constant variance, the Fisher information can decrease for unknown systematic errors and, thus, the Cramér-Rao bound can increase with an increasing measurement time. For the typically existing case of simultaneously occurring random and unknown systematic errors, an optimal measurement time exists for which the achievable measurement uncertainty becomes minimal. In summary, the examples demonstrate how to determine the Fisher information and the Cramér-Rao bound for unknown systematic errors.

1. Introduction

In metrology, the identification and understanding of measurement uncertainty limits is an important task, e.g., for optimizing the current setup of a measurement system or for designing an improved measurement system. For this purpose, the Cramér-Rao inequality [1,2] is well-known as a valuable tool and has been applied successfully for many different investigations, e.g., for ultrasound displacement sensors [3], sonar and radar systems [4], optical flow velocity measurements [5–10] or optical distance and shape measurements [11–13]. The Cramér-Rao inequality allows to determine the minimal achievable variance for unbiased and biased estimations [14,15]. This variance limit is known as the Cramér-Rao bound. Considering a single unknown quantity θ, the Cramér-Rao bound for every unbiased estimator $\hat{\theta}$ is the reciprocal of the Fisher information $\mathscr{I}_\theta$, i.e., [16–18]

$$\mathrm{Var}(\hat{\theta}) \geqslant \frac{1}{\mathscr{I}_\theta}. \tag{1}$$

Hence, the Fisher information is the crucial parameter for understanding the minimal achievable measurement uncertainty.

The concept of Fisher information is based on a (known) probability density function referred to as the likelihood function of the signal to be evaluated. For this reason, the concept is applicable for random measurement errors. Many studies considered enhancements, e.g., regarding the estimation of multiple unknown quantities with constraints [19–21], the estimation of complex parameters [22] and the case of a singular Fisher information matrix [23–25]. However, the explicit treatment of systematic errors such as an unknown offset or linear drift in the context of the Fisher information is not yet clear.

Especially for an unknown offset, one approach might be to model the unknown systematic error as an additional unknown quantity with constraints. An alternative approach that is easily applicable for all kinds of systematic errors directly follows from the international "Guide to the expression of uncertainty in measurement" (GUM) [26,27]. The GUM suggests to correct known systematic errors and to treat remaining unknown systematic errors the same way as random errors after assigning an appropriate probability density function using the principle of maximum entropy [28]. According to this suggestion, the unknown systematic errors can also be treated within the framework of the Fisher information, which remains to be investigated.

At first sight, information is assumed to increase with an increasing measurement time, because once acquired the information cannot disappear. On the other hand, it is well-known that the variance of the measurement decreases with the measurement time in case of dominant

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random errors but can increase in case of dominant systematic errors. One prominent example for such a behavior of the measurement uncertainty versus the measurement time is the time measurement or the frequency stability of clocks, e.g., due to temperature drifts [29]. Consequently, unknown systematic errors are assumed to cause a decrease in information (destruction of the information) with an increasing measurement time. However, the attribution of a specific information decrease to a certain kind of systematic error and vice versa is not yet clear.

The aim of the article is to demonstrate that the Fisher information and, thus, the Cramér-Rao bound of a measurand can be determined for unknown systematic errors. After presenting the state-of-the-art case of a random error namely additive Gaussian noise in Section 2, the treatment of systematic errors is discussed in Section 3 considering three different examples. For each example, the corresponding characteristic behavior of the Fisher information and the Cramér-Rao bound with respect to the measurement time is derived. Finally, an example for the superposition of random and systematic errors is discussed in Section 4, which occurs in real measurements in general.

Throughout the article, the case of a single unknown constant is considered. This case applies frequently and is easy to calculate, which helps focusing on the key aspects of the article. For the same reason, the disturbances are always modeled by normal distributions, although the calculations can also be performed for other probability density functions. Since the subsequent findings are not restricted to a specific measurement quantity, the unknown constant is normalized with respect to its unit.

2. Random errors

As a reference for the subsequent investigations, the case of an unknown constant θ superposed by additive Gaussian noise with the variance σ^2 is considered. The acquired signal x thus reads

$$x = \theta + w, \qquad \text{with} \quad w \sim \mathcal{N}(0,\sigma^2). \tag{2}$$

As a result, the likelihood function $p(x,\theta)$ is a normal distribution $\mathcal{N}(\theta,\sigma^2)$ with the mean value θ and the variance σ^2. The first derivative $\frac{\partial \ln p(x,\theta)}{\partial\theta}$ of the log-likelihood function is named score function. Using the second derivative of the log-likelihood function, the Fisher information is [17, p. 111]

$$\mathcal{I}_\theta = -\mathrm{E}\left(\left(\frac{\partial^2 \ln p(x,\theta)}{\partial\theta^2}\right)^2\right) = \frac{1}{\sigma^2} \tag{3}$$

and, according to Eq. (1), σ^2 results as the Cramér-Rao bound of θ.

Note that the noise variance is often unknown, which means that two unknown parameters exist and the calculated Fisher information becomes one of the four elements of the Fisher information matrix. However, the same Cramér-Rao bound of θ results, which follows from the respective element of the main diagonal of the inverse Fisher information matrix [17, p. 112]. For this reason, the consideration of θ as the only unknown parameter is a convenient simplification.

For white noise with the constant noise power spectral density S, the noise variance depends on the bandwidth of the measurement or on the measurement time T, respectively, according to the relation $\sigma^2 = S/T$. Inserting this relation into Eq. (3), the Fisher information becomes

$$\mathcal{I}_\theta = \frac{1}{S}\cdot T. \tag{4}$$

As a result, the Fisher information is directly proportional to the measurement time.

For a measurement, which starts at the time $t = 0$, the calculated characteristic behavior of the Fisher information over the time t is depicted in Fig. 1 for the white noise condition. The corresponding temporal behavior of the Cramér-Rao bound is shown in addition resulting from the reciprocal of the Fisher information. Before the measurement starts, the information is zero and the measurement uncertainty is therefore infinite (no previous knowledge). The longer the measurement time the more information is obtained and, thus, the lower the Cramér-Rao bound. For $t \to \infty$, the Fisher information converges to infinity and the Cramér-Rao bound converges to zero.

For colored noise, the increase of the Fisher information can be smaller or larger than for the white noise condition, and the slope of the Fisher information can vary between zero and plus infinity. For instance, the initial slope of the Fisher information is larger/smaller for a noise power spectral density with a high-pass/low-pass characteristic than for white noise.

3. Systematic errors

The definition of the Fisher information contains so-called regularity conditions with respect to the likelihood function [17, p. 111]. The first regularity condition is the existence of the score function, i.e., the first derivative of the log-likelihood function. The second condition is that the order of the integration with respect to x and the differentiation with respect to θ can be interchanged in the expression $\frac{\partial}{\partial\theta}\int p(x,\theta)\mathrm{d}x$. The likelihood function used in Section 2 fulfills these requirements. When the regularity conditions hold, the mean of the score function $\frac{\partial \ln p(x,\theta)}{\partial\theta}$ is zero, i.e., $\mathrm{E}\left(\frac{\partial \ln p(x,\theta)}{\partial\theta}\right) = 0$ [17, p. 112].

For a known systematic measurement error, the likelihood function $p(x,\theta)$ is a Dirac distribution that is located at the value of the error. Since $\mathrm{E}\left(\frac{\partial \ln p(x,\theta)}{\partial\theta}\right) \neq 0$, the regularity conditions are not fulfilled for known systematic errors. As a result, the concept of Fisher information is not applicable to known systematic errors. Fortunately this is meaningless, because the international guide to the expression of uncertainty in measurement demands the correction of known systematic errors.

In order to determine the Fisher information for an unknown systematic error, an appropriate probability density function is required, which describes the error behavior and simultaneously fulfills the regularity conditions. The subsequently applied normal distributions with zero mean and a variance that is independent of the unknown quantity fulfill the regularity conditions [17]. Furthermore, the modeling of a systematic error as a random error is in perfect agreement with the international guide to the expression of uncertainty in measurement. Three frequent kinds of systematic errors are discussed in the following subsections.

3.1. Offset

Measurements often contain an unknown offset c_0, i.e., the measured value deviates from the true value and the deviation does not vary. As a result, the acquired signal for measuring the unknown constant θ reads

$$x = \theta + c_0 \tag{5}$$

independent of the measurement time. The origin of the offset can be, e.g., an inaccurate calibration or the aging of certain components of the measurement system.

Since the offset is an unknown systematic error, it is modeled as a purely random error with zero mean. Note that the offset is random when the measurement starts, but it does not vary with time during the measurement. Hence, the offset c_0 is not a random variable, but the realization of a random variable. Here, the random behavior is described by a normal distribution $\mathcal{N}(0,\sigma_0^2)$ with zero mean and the constant variance σ_0^2 as the probability density function. As a result, the Fisher information that follows from Eq. (3) is independent of the measurement time and amounts to

$$\mathcal{I}_\theta = \frac{1}{\sigma_0^2}. \tag{6}$$

Consequently, the Cramér-Rao lower bound is also independent of the measurement time.

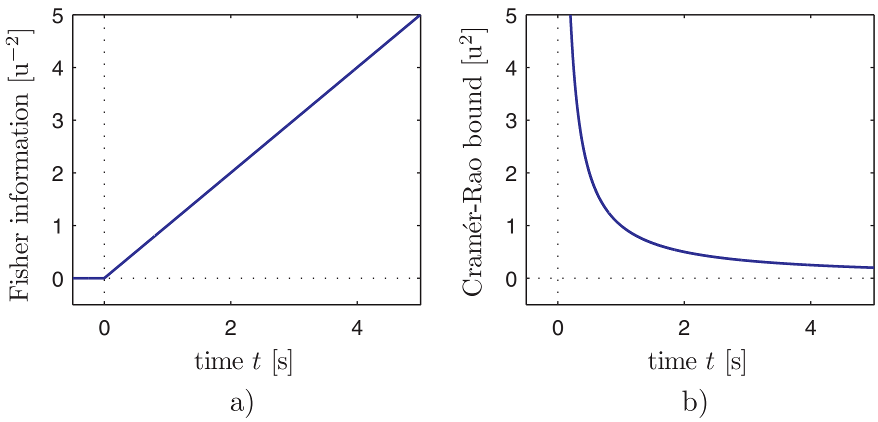

Fig. 1. (a) Fisher information and (b) Cramér-Rao bound versus time for the measurement of an unknown constant quantity θ disturbed by additive white Gaussian noise. The measurement starts at $t = 0$, the arbitrary unit of θ is u, and the noise power spectral density amounts to = 1 u^2/Hz.

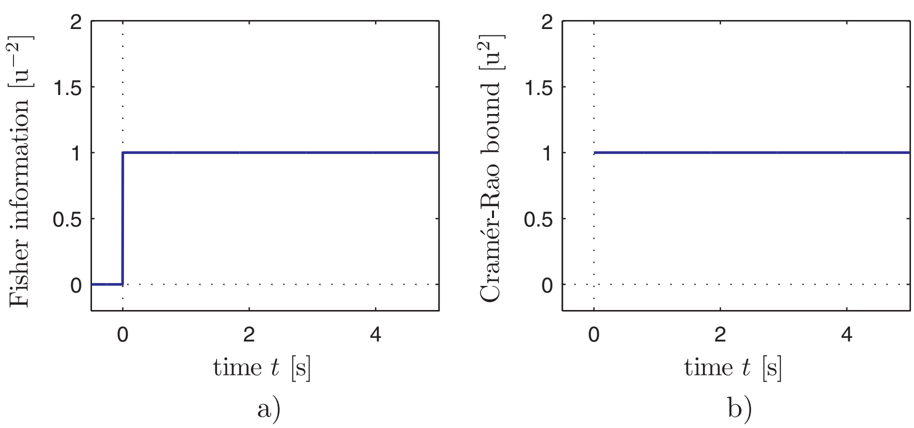

Fig. 2. (a) Fisher information and (b) Cramér-Rao bound versus time for the measurement of an unknown constant quantity θ superposed by an unknown offset. The measurement starts at $t = 0$, the arbitrary unit of θ is u, and $\sigma_0 = 1$ u.

The temporal behavior of the Fisher information for a measurement starting at $t = 0$ is depicted in Fig. 2 together with the corresponding Cramér-Rao bound, i.e., the reciprocal of the Fisher information. With the start of the measurement, the Fisher information instantly jumps to the value $\frac{1}{\sigma_0^2}$ and remains there. The qualitative behavior of the Cramér-Rao bound is similar. However it jumps from plus infinity to σ_0^2 and remains there despite an increasing measurement time.

3.2. Linear drift

If an integration is part of the measurement, an unknown offset at the integrator input signal leads to a linearly increasing error of the integrator output. This is one example for a linear drift, which applies for instance when estimating the distance to an object by integrating the detected object velocity with respect to the time. A similar example is the time measurement by counting clock pulses, i. e., the occurrence of equally spaced events. Another reason of a linear drift could be an existing cross-sensitivity and the variation of the respective influence quantity. Note that systematic errors occur not necessarily due to cross-sensitivities. A systematic error can also result from a real temporal variation of the measurand during the measurement time, while the measurement aims to measure a constant quantity only.

The acquired signal as a function of the measurement time T reads for an unknown constant θ (i.e., the true value of the integrator input is zero) superposed by a linearly increasing error with an unknown constant slope c_1

$$x(T) = \theta + c_1 \cdot T. \tag{7}$$

Similar to the previous example of the offset, the parameter c_1 is now treated as the realization of a random variable. For convenience, this random variable is assumed to follow a normal distribution $\mathcal{N}(0,\sigma_1^2)$ with zero mean and the variance σ_1^2. As a result, the value of $c_1 T$ has the variance $\sigma_1^2 T^2$ and, thus, the Fisher information reads according to Eq. (3)

$$\mathcal{I}_\theta = \frac{1}{\sigma_1^2} \cdot \frac{1}{T^2}. \tag{8}$$

Hence, the Fisher information is indirectly proportional to the square of the measurement time. In order words, the Fisher information decreases with an increasing measurement time when considering an unknown linear drift. This finding proves the hypothesis that unknown systematic errors can lead to a decreasing Fisher information.

In order to illustrate the finding, the calculated temporal behavior of the Fisher information is depicted in Fig. 3 together with the corresponding Cramér-Rao bound, i.e., the reciprocal of the Fisher information. At the measurement start at $t = 0$, the Fisher information instantly jumps from zero to plus infinity and then converges to zero for $t \to \infty$. The Cramér-Rao bound is zero at $t = 0$ and increases with the square of the measurement time. As a result, the lower the measurement time the lower the Cramér-Rao bound.

3.3. Successive drifts or the adding of successive errors

A different temporal behavior of the Fisher information results from a variant of the previous example. If the error of the integrator input signal changes after a certain duration, a sequence of unknown linear drifts occurs. For instance, this condition applies for a numerical integration (or summation) of a discrete-time signal, where each sample contains a different error. The respective errors of the integrator output add to each other with each integration (or summation) step.

The discrete signal for an unknown constant quantity θ then reads at the n-th time step

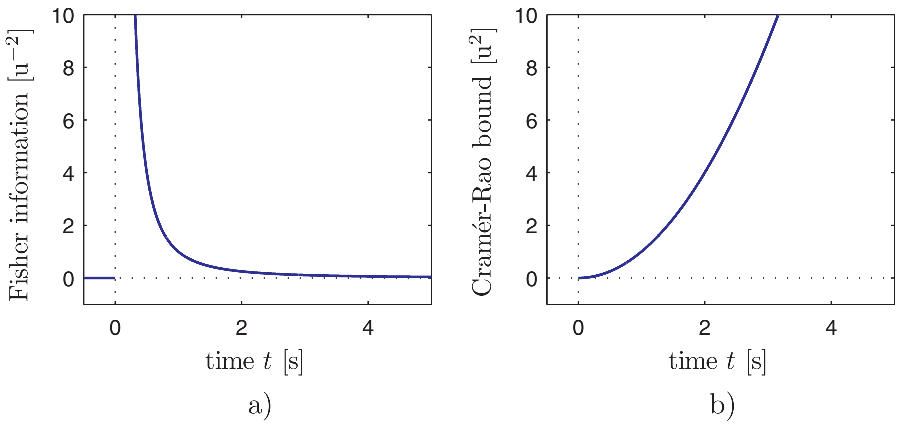

Fig. 3. (a) Fisher information and (b) Cramér-Rao bound versus time for the measurement of an unknown constant quantity θ disturbed by an unknown linear drift. The measurement starts at $t = 0$, the arbitrary unit of θ is u, and $\sigma_1 = 1$ u/s.

$$x[n] = \theta + \sum_{i=1}^{n} c_i \tau, \qquad n \in \mathbb{N}, \tag{9}$$

where τ is the sampling period or the step size of the discrete integration. Treating the unknown slopes c_i of the successive linear drifts as uncorrelated realizations of a common random variable with the normal distribution $\mathcal{N}(0,\tilde{\sigma}_1^2)$, the variance of each error $c_i\tau$ amounts to $\tilde{\sigma}_1^2\tau^2$. Since the variance of the sum of the uncorrelated errors equals the sum of all variances, the variance of the total error is $n\tilde{\sigma}_1^2\tau^2 = \tilde{\sigma}_1^2\tau T$ with the measurement time $T = n{\cdot}\tau$. Furthermore, the sum of normally distributed errors follows a normal distribution as well. Hence, inserting the variance $\tilde{\sigma}_1^2\tau T$ into Eq. (3) finally gives the Fisher information

$$\mathcal{I}_\theta = \frac{1}{\tilde{\sigma}_1^2\tau}\cdot\frac{1}{T}. \tag{10}$$

The Fisher information is indirectly proportional to the measurement time, which is a further example of an unknown systematic error that results in a decreasing Fisher information.

For the sake of completeness, the temporal behavior of the Fisher information is depicted in Fig. 4 together with the corresponding Cramér-Rao bound for a measurement start at $t = 0$. Aside from the different relations with respect to the measurement time, the general tendencies of the curves are the same as in Fig. 3.

4. Superposed random and unknown systematic errors

In order to demonstrate the more realistic case of superposed random and unknown systematic errors, the combination of additive white Gaussian, an unknown offset and an unknown linear drift is considered:

$$x(T) = \theta + w + c_0 + c_1{\cdot}T. \tag{11}$$

Since the three errors are uncorrelated and obey a normal distribution here (cf. Sections 2 and 3), the sum of the three errors obeys a normal distribution as well. The mean of the summed error is zero and the variance amounts to $S/T + \sigma_0^2 + \sigma_1^2 T^2$. According to Eq. (3), the Fisher information as a function of the measurement time T thus takes the form

$$\mathcal{I}_\theta = \frac{1}{\frac{S}{T} + \sigma_0^2 + \sigma_1^2 T^2}. \tag{12}$$

Assuming the noise power spectral density $S = 0.5\ \mathrm{u}^2/\mathrm{Hz}$, the offset standard deviation $\sigma_0 = 1$ u and the drift standard deviation $\sigma_1 = 0.01$ u/s, the calculated temporal behavior of the Fisher information and the Cramér-Rao bound is shown in Fig. 5. The red crosses indicate the variance of the measured values obtained from a Monte-Carlo simulation. The simulation results agree with the analytical calculation of the Cramér-Rao bound, which verifies the calculated curve. The dashed lines represent the results for each single error. As a result, the additive white Gaussian noise dominates for a short measurement time, whereas the offset and later the drift mainly limit the Fisher information and the Cramér-Rao bound, respectively, with an increasing measurement time. Regarding the Fisher information, each result from a single error represents an upper bound. In contrast to this, each Cramér-Rao bound from a single error is a lower bound of the total Cramér-Rao bound. Note that a maximum occurs for the Fisher information, which leads to a minimum of the Cramér-Rao bound. As a result, the measurement uncertainty limit is minimal for an appropriate choice of the measurement time.

5. Conclusions

The concept of Fisher information and the Cramér-Rao bound is not only applicable to random errors but also unknown systematic errors. This insight follows from the international guide to the expression of uncertainty in measurement (GUM), where the behavior of the unknown systematic error shall be described by a random process with an

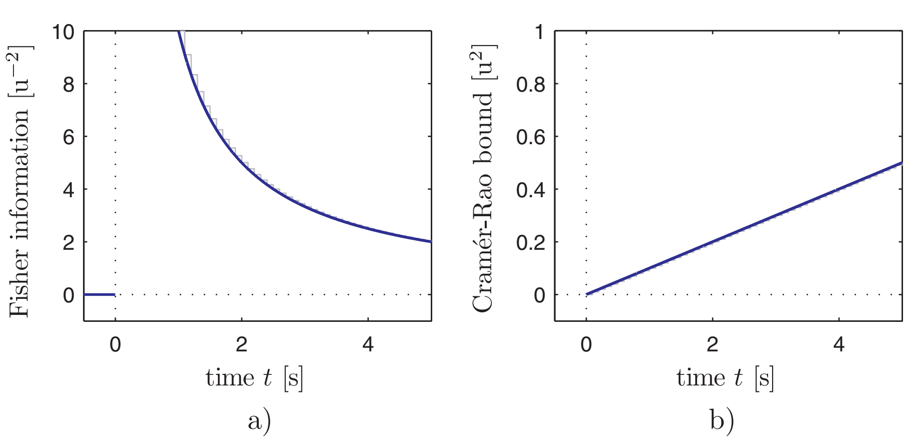

Fig. 4. (a) Fisher information and (b) Cramér-Rao bound versus time for the measurement of an unknown constant quantity θ disturbed by successive unknown linear drifts. The measurement starts at $t = 0$, the arbitrary unit of θ is u, $\sigma_1 = 1$ u/s and $\tau = 0.1$ s.

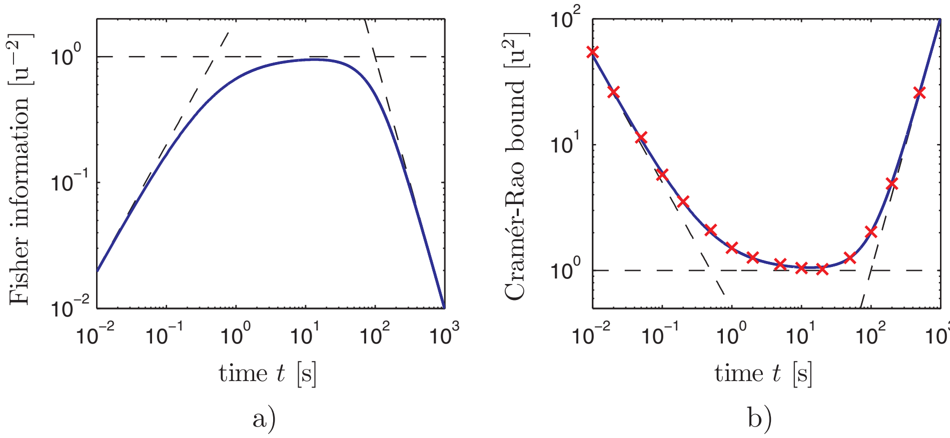

Fig. 5. (a) Fisher information and (b) Cramér-Rao bound versus time for the measurement of an unknown constant quantity θ superposed by additive white Gaussian noise and two unknown systematic errors (offset and linear drift). The measurement started at $t = 0$. The arbitrary unit of θ is u, $S = 0.5\ \mathrm{u}^2/\mathrm{Hz}$, $\sigma_0 = 1\ \mathrm{u}$, $\sigma_1 = 0.01\ \mathrm{u/s}$. A Monte-Carlo simulation (red crosses) verifies the calculated Cramér-Rao bound. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 1
Overview of the behavior of the Fisher information $\mathscr{I}_\theta$ with respect to the measurement time $T > 0$ for the investigated errors. Note that the Cramér-Rao bound for every unbiased estimator is $\mathscr{I}_\theta^{-1}$.

$\mathscr{I}_\theta$	$\frac{\partial \mathscr{I}_\theta}{\partial T}$	error type	source
(Increase above $\propto T$)	>0	Random	Additive colored Gaussian noise (high-pass)
$\propto T$	>0	Random	Additive white Gaussian noise
(increase below $\propto T$)	≥ 0	Random	Additive colored Gaussian noise (low-pass)
$=$ const	$=0$	Systematic	Offset
$\propto \frac{1}{T}$	<0	Systematic	Successive linear drifts
$\propto \frac{1}{T^2}$	<0	Systematic	Linear drift

appropriate probability density function.

In the present article, the behavior of the Fisher information and the Cramér-Rao bound with an increasing measurement time were derived for an unknown constant measurand regarding a random error (additive Gaussian noise) and three unknown systematic errors (offset, linear drift, successive linear drifts). While a random error is usually described by a stationary process, the unknown systematic errors cover the broader class of non-stationary processes. As a result, the increase of the Fisher information with respect to the measurement time is always positive or zero for additive Gaussian noise but is negative or zero for the considered unknown systematic errors. In particular, the Fisher information as a function of the measurement time is shown to be constant for an offset, to be indirectly proportional to the square of the measurement time for a linear drift and to be indirectly proportional to the measurement time for successive linear drifts. These findings are summarized in Table 1. Note that, although unknown systematic errors can lead to a decrease of the Fisher information, the Fisher information itself is always positive or zero as expected. Hence, the slope of the Fisher information with respect to the time is in general between minus and plus infinity, whereas the Fisher information is between zero and plus infinity.

According to the Cramér-Rao bound, which is the reciprocal of the derived Fisher information, a lower bound of the achievable measurement uncertainty is obtained for the unknown systematic errors. In contrast to added noise with a constant variance, the Cramér-Rao bounds resulting from unknown systematic errors increase or stay constant with an increasing measurement time. For this reason, an optimal measurement time can exist for which the Cramér-Rao bound is minimal, if random and unknown systematic errors occur simultaneously.

The investigations illustrate the significance of the measurement time as a crucial parameter of measurements. As a conclusion, the measurement uncertainty should always be noted together with the measurement time.

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